$(x+dx+u_x+du_x,\ y+dy+u_y+du_y,\ z+dz+u_z+du_z)$.

Now

$$u_x+du_x = u_x + \frac{\partial u_x}{\partial x}dx + \frac{\partial u_x}{\partial y}dy + \frac{\partial u_x}{\partial z}.dz$$

$$u_y+du_y = u_y + \frac{\partial u_y}{\partial x}dx + \frac{\partial u_y}{\partial y}dy + \frac{\partial u_y}{\partial z}.dz$$

$$u_z+du_z = u_z + \frac{\partial u_z}{\partial x}dx + \frac{\partial u_z}{\partial y}dy + \frac{\partial u_z}{\partial z}dz$$

The square of the distance between P' and Q' is

$$ds'^2 = \left[\left(1+\frac{\partial u_x}{\partial x}\right)dx + \frac{\partial u_x}{\partial y}dy + \frac{\partial u_x}{\partial z}dz\right]^2$$
$$+\left[\frac{\partial u_y}{\partial x}dx + \left(1+\frac{\partial u_y}{\partial y}\right)dy + \frac{\partial u_y}{\partial z}dz\right]^2$$
$$+\left[\frac{\partial u_z}{\partial x}dx + \frac{\partial u_z}{\partial y}dy + \left(1+\frac{\partial u_z}{\partial z}\right)dz\right]^2$$

$^2C_3^n$

$$= \left(1+2\frac{\partial u_x}{\partial x}\right)(dx)^2 + 2\frac{\partial u_x}{\partial y}dx\,dy + 2\frac{\partial u_x}{\partial z}dx\,dz$$
$$+\left(1+2\frac{\partial u_y}{\partial y}\right)(dy)^2 + 2\frac{\partial u_y}{\partial x}dy\,dx + 2\frac{\partial u_y}{\partial z}dy\,dz$$
$$+\left(1+2\frac{\partial u_z}{\partial z}\right)(dz)^2 + 2\frac{\partial u_z}{\partial x}dz\,dx + 2\frac{\partial u_z}{\partial y}dz\,dy$$

[neglecting squares and products of partial derivatives of u_x, u_y, u_z w.r.t. ds'^2]

$$\therefore ds'^2 - ds^2 = \left[2\frac{\partial u_x}{\partial x}(dx)^2 + 2\frac{\partial u_y}{\partial y}(dy)^2 + 2\frac{\partial u_z}{\partial z}(dz)^2\right.$$
$$+2\left(\frac{\partial u_z}{\partial y}+\frac{\partial u_y}{\partial z}\right)dy\,dz + 2\left(\frac{\partial u_z}{\partial x}+\frac{\partial u_x}{\partial z}\right)dz\,dx$$
$$\left.+2\left(\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right)dx\,dy\right]$$

Dividing by ds^2 and proceeding to the limit $ds\to 0$ and $ds'=ds$ where $\epsilon\to 0$ as $ds\to 0$, we get using $ds \simeq ds'$ for small deformation

$$\lim_{ds\to 0}\frac{ds'+ds}{ds}\cdot\frac{ds'-ds}{ds} \simeq 2\lim_{ds\to 0}\frac{ds'-ds}{ds}$$
$$= 2\left(e_{xx}l^2 + e_{yy}m^2 + e_{zz}n^2 + 2e_{yz}mn + 2e_{zx}nl + 2e_{xy}lm\right)$$

$$\therefore \lim_{ds\to 0}\frac{ds'-ds}{ds} = \left(e_{xx}l^2 + e_{yy}m^2 + e_{zz}n^2 + 2e_{yz}mn + 2e_{zx}nl + 2e_{xy}lm\right)$$

L.H.S. is recognised as the increment in length per unit length in the direction (l,m,n) at (x,y,z)

Thus $e_{xx}l^2 + e_{yy}m^2 + e_{zz}n^2 + 2e_{yz}mn + 2e_{zx}nl + 2e_{xy}lm$ is the increment in length per unit length at (x,y,z) in the direction (l,m,n).

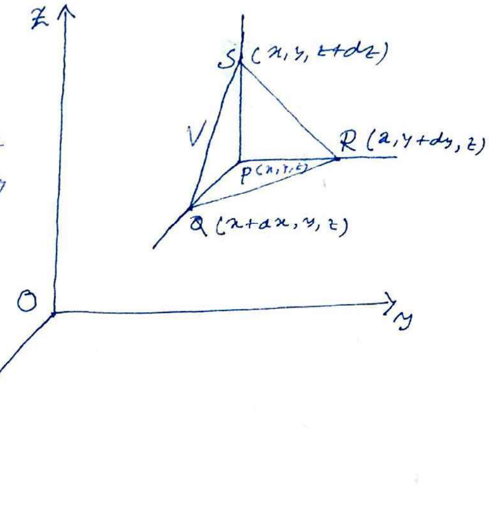
If $l=1, m=0, n=0$ then increment in length per unit length at (x,y,z) in x-direction $= e_{xx}$

Similarly, $e_{yy} =$ increment in length per unit length in y-direction where $l=0, m=1, n=0$

& $e_{zz} =$ increment in length per unit length in z-direction where $l=0, m=0, n=1$

98) VOLUMETRIC STRAIN OR CUBICAL DILATATION

In the undeformed state of material let us consider a tetrahedron PQRS of material occupying a small volume V within the material. such that the co-ordinates of P, Q, R, S are respectively (x,y,z), $(x+dx,y,z)$, $(x,y+dy,dz)$, $(x,y,z+dz)$.



After small deformation, P occupies the position $(x+u_x, y+u_y, z+u_z)$ where u_x, u_y, u_z are the components of displacement of the point $P(x,y,z)$.

The co-ordinates of Q after deformation is

$$= \left(x+dx+u_x+\frac{\partial u_x}{\partial x}dx,\ y+u_y+\frac{\partial u_y}{\partial x}dx,\ z+u_z+\frac{\partial u_z}{\partial x}dx\right)$$

and those of R in the deformed state is

$$= \left(x+u_x+\frac{\partial u_x}{\partial y}dy,\ y+dy+u_y+\frac{\partial u_y}{\partial y}dy,\ z+u_z+\frac{\partial u_z}{\partial y}dy\right)$$

and of S in deformed state is

$$= \left(x+u_x+\frac{\partial u_x}{\partial z}dz,\ y+u_y+\frac{\partial u_y}{\partial z}dz,\ z+dz+u_z+\frac{\partial u_z}{\partial z}dz\right)$$

So after deformation the volume of tetrahedron becomes

$$V' = \frac{1}{6}\begin{vmatrix} x+u_x & y+u_y & z+u_z & 1 \\ x+dx+u_x+\frac{\partial u_x}{\partial x}dx & y+u_y+\frac{\partial u_y}{\partial x}dx & z+u_z+\frac{\partial u_z}{\partial x}dx & 1 \\ x+u_x+\frac{\partial u_x}{\partial y}dy & y+dy+u_y+\frac{\partial u_y}{\partial y}dy & z+u_z+\frac{\partial u_z}{\partial y}dy & 1 \\ x+u_x+\frac{\partial u_x}{\partial z}dz & y+u_y+\frac{\partial u_y}{\partial z}dz & z+dz+u_z+\frac{\partial u_z}{\partial z}dz & 1 \end{vmatrix}$$

or,

$$V' = \frac{1}{6}\begin{vmatrix} x+u_x & y+u_y & z+u_z & 1 \\ \left(1+\frac{\partial u_x}{\partial x}\right)dx & \frac{\partial u_y}{\partial x}\cdot dx & \frac{\partial u_z}{\partial x}dx & 0 \\ \frac{\partial u_x}{\partial y}dy & \left(1+\frac{\partial u_y}{\partial y}\right)dy & \frac{\partial u_z}{\partial y}dy & 0 \\ \frac{\partial u_x}{\partial z}\cdot dz & \frac{\partial u_y}{\partial z}\cdot dz & \left(1+\frac{\partial u_z}{\partial z}\right)dz & 0 \end{vmatrix}$$

$$= \frac{1}{6}\left(1+\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}+\frac{\partial u_z}{\partial z}\right)dx\,dy\,dz$$

Assuming that the strains are small quantities such that their squares and products can be neglected.

Before deformation the volume of the tetrahedron is

$$V = \frac{1}{6}dx\,dy\,dz$$

$$\therefore \Delta V = V' - V = \frac{1}{6}\left(\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}+\frac{\partial u_z}{\partial z}\right)dx\,dy\,dz$$

$$= V\,(e_{xx}+e_{yy}+e_{zz})$$

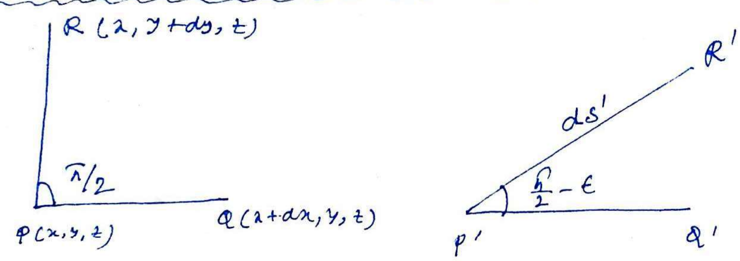
$\therefore$ increment in volume per unit volume $= \frac{\Delta V}{V} = e_{xx}+e_{yy}+e_{zz}$

This is called the volumetric strain or cubical dilatation.

Thus volumetric strain or cubical dilatation equals to sum of the three linear strains.

GEOMETRICAL INTERPRETATION OF SHEARING ~~STRESS~~ strains

994 991



Consider three points P, Q, R in the undeformed state of the material such that the co-ordinates of P, Q and R are respectively (x,y,z), $(x+dx, y, z)$, $(x, y+dy, z)$. The length of $PQ = dx$ and that of $PR = dy$ and the angle between PQ and PR is $\frac{\pi}{2}$.

For small deformation P occupies the position P' with co-ordinates $(x+u_x, y+u_y, z+u_z)$ and Q takes the position whose co-ordinates are

$$\left(x+dx+u_x+\frac{\partial u_x}{\partial x}dx,\ y+u_y+\frac{\partial u_y}{\partial x}dx,\ z+u_z+\frac{\partial u_z}{\partial x}dx\right)$$

and that of R' which is the new position of R

$$= \left(x+u_x+\frac{\partial u_x}{\partial y}dy,\ y+dy+u_y+\frac{\partial u_y}{\partial y}dy,\ z+u_z+\frac{\partial u_z}{\partial y}dy\right)$$

If $P'Q' = ds$ and $P'R' = ds'$
then the d-cs of $P'Q'$ and $P'R'$ are respectively

$$\left[\left(1+\frac{\partial u_x}{\partial x}\right)\frac{dx}{ds};\ \frac{\partial u_y}{\partial x}\cdot\frac{dx}{ds};\ \frac{\partial u_z}{\partial x}\cdot\frac{dx}{ds}\right],\ \left[\frac{\partial u_x}{\partial y}\cdot\frac{dy}{ds'};\ \left(1+\frac{\partial u_y}{\partial y}\right)\frac{dy}{ds'};\ \frac{\partial u_z}{\partial y}\cdot\frac{dy}{ds'}\right]$$

$$\therefore P'Q' = ds = \sqrt{\left(1+\frac{\partial u_x}{\partial x}\right)^2 (dx)^2 + \left(\frac{\partial u_y}{\partial x}\right)^2 (dx)^2 + \left(\frac{\partial u_z}{\partial x}\right)^2 (dx)^2}$$

$$\simeq \left(1+\frac{\partial u_x}{\partial x}\right)dx$$

Neglecting second and higher order quantities $\left[\left(\frac{\partial u_x}{\partial x}\right)^2, \left(\frac{\partial u_x}{\partial x}\right)^3, \ldots\right]$

$$P'R' = ds' = \sqrt{\left(\frac{\partial u_x}{\partial y}\right)^2 (dy)^2 + \left(1+\frac{\partial u_y}{\partial y}\right)^2 (dy)^2 + \left(\frac{\partial u_z}{\partial y}\right)^2 (dy)^2}$$

$$\simeq \left(1+\frac{\partial u_y}{\partial y}\right)dy$$

$$\therefore \frac{ds}{dx} = 1 + \frac{\partial u_x}{\partial x} \qquad \frac{dx}{ds} = \frac{1}{1+\frac{\partial u_x}{\partial x}}, \quad \frac{dy}{ds'} = \frac{1}{1+\frac{\partial u_y}{\partial y}}$$

so d-cs of $P'Q'$ and $P'R'$ are in the limit

$$1, \frac{\partial u_y}{\partial x}, \frac{\partial u_z}{\partial x} \quad \text{and} \quad \frac{\partial u_x}{\partial y}, 1, \frac{\partial u_z}{\partial y}$$

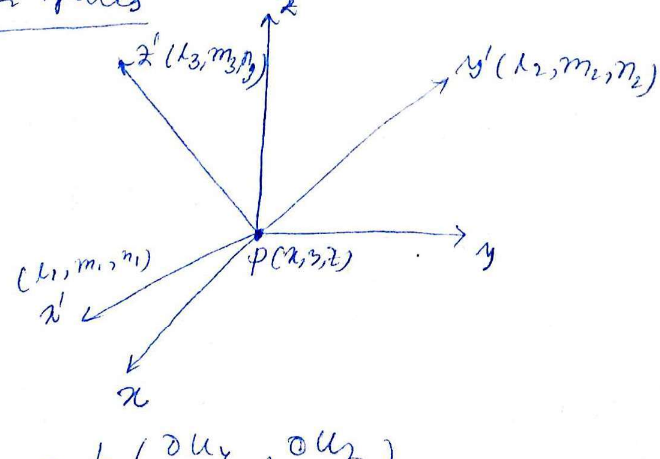
If the angle between $P'Q'$ and $P'R'$ be $\frac{\pi}{2} - \epsilon$, then

$$\cos\left(\frac{\pi}{2}-\epsilon\right) = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \quad \text{(neglecting second order quantity)}$$

$$= 2e_{xy}$$

similar interpretation can be given $2e_{zx}$ and $2e_{yz}$

29. Strain transformation rules



If the displacement components u_x, u_y, u_z at a point $P(x,y,z)$ are known functions of x,y,z then six strain components

$$e_{xx} = \frac{\partial u_x}{\partial x}, \quad e_{yy} = \frac{\partial u_y}{\partial y}$$

$$e_{zz} = \frac{\partial u_z}{\partial z},$$

$$e_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right), \quad e_{yz} = \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right)$$

$$e_{zx} = \frac{1}{2}\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right)$$

can be determined with reference to three perpendicular axes Ox, Oy, Oz through P.

At O we take another set of rectangular axes Ox', Oy', Oz' with d-cs $(l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3)$. The law of transformation of co-ordinates from the set of axes to another are given by the following scheme

	x	y	z
x'	l_1	m_1	n_1
y'	l_2	m_2	n_2
z'	l_3	m_3	n_3

The components of strain with reference to new set of axes Ox', Oy', Oz' are $e_{x'x'}, e_{y'y'}, e_{z'z'}, e_{x'y'}, e_{y'z'}, e_{z'x'}$.

Now

$$e_{x'x'} = \frac{\partial u_{x'}}{\partial x'} = \frac{\partial u_{x'}}{\partial x}\cdot\frac{\partial x}{\partial x'} + \frac{\partial u_{x'}}{\partial y}\cdot\frac{\partial y}{\partial x'} + \frac{\partial u_{x'}}{\partial z}\cdot\frac{\partial z}{\partial x'}$$

$$= \left(l_1\frac{\partial}{\partial x} + m_1\frac{\partial}{\partial y} + n_1\frac{\partial}{\partial z}\right)u_{x'}$$

$$= \left(l_1\frac{\partial}{\partial x} + m_1\frac{\partial}{\partial y} + n_1\frac{\partial}{\partial z}\right)(l_1 u_x + m_1 u_y + n_1 u_z)$$

$$= e_{xx}l_1^2 + e_{yy}m_1^2 + e_{zz}n_1^2 + m_1n_1\left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right) + n_1l_1\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) + l_1m_1\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right)$$

$$\therefore e_{x'x'} = e_{xx}l_1^2 + e_{yy}m_1^2 + e_{zz}n_1^2 + 2l_1m_1e_{xy} + 2m_1n_1e_{yz} + 2n_1l_1e_{zx}$$

similarly,

$$e_{y'y'} = e_{xx}l_2^2 + e_{yy}m_2^2 + e_{zz}n_2^2 + 2l_2m_2e_{xy} + 2m_2n_2e_{yz} + 2n_2l_2e_{zx}$$

$$e_{z'z'} = e_{xx}l_3^2 + e_{yy}m_3^2 + e_{zz}n_3^2 + 2l_3m_3e_{xy} + 2m_3n_3e_{yz} + 2n_3l_3e_{zx}$$

Now

$$e_{x'y'} = \frac{1}{2}\left(\frac{\partial u_{x'}}{\partial y'} + \frac{\partial u_{y'}}{\partial x'}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x}\frac{\partial x}{\partial y'} + \frac{\partial}{\partial y}\frac{\partial y}{\partial y'} + \frac{\partial}{\partial z}\cdot\frac{\partial z}{\partial y'}\right)u_{x'} + \frac{1}{2}\left(\frac{\partial}{\partial x}\cdot\frac{\partial x}{\partial x'} + \frac{\partial}{\partial y}\cdot\frac{\partial y}{\partial x'} + \frac{\partial}{\partial z}\cdot\frac{\partial z}{\partial x'}\right)u_{y'}$$

$$= \frac{1}{2}\left(l_2\frac{\partial}{\partial x} + m_2\frac{\partial}{\partial y} + n_2\frac{\partial}{\partial z}\right)(l_1u_x + m_1u_y + n_1u_z) + \frac{1}{2}\left(l_1\frac{\partial}{\partial x} + m_1\frac{\partial}{\partial y} + n_1\frac{\partial}{\partial z}\right)(l_2u_x + m_2u_y + n_2u_z)$$

$$= l_1l_2\frac{\partial u_x}{\partial x} + m_1m_2\frac{\partial u_y}{\partial y} + n_1n_2\frac{\partial u_z}{\partial z} + \frac{1}{2}(m_2n_1 + m_1n_2)\left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right) + \frac{1}{2}(n_2l_1 + n_1l_2)\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right) + \frac{1}{2}(l_2m_1 + l_1m_2)\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$$

$$= l_1l_2e_{xx} + m_1m_2e_{yy} + n_1n_2e_{zz} + (m_2n_1 + m_1n_2)e_{yz} + (n_2l_1 + n_1l_2)e_{zx} + (l_2m_1 + l_1m_2)e_{xy}$$

similarly $e_{z'x'}$ and $e_{y'z'}$ can be determined. These are strain transformation rules.

~~Maximum~~ strains and strain direction

Principal > when the displacement field is defined at a point $P(x,y,z)$, the extension in length per unit length in the direction PQ (say) where the d-cs of PQ is l,m,n; is given by, $e_{PQ} = e_{xx}l^2 + e_{yy}m^2 + e_{zz}n^2 + 2e_{yz}mn + 2e_{zx}nl + 2e_{xy}lm$ ... (1)

obviously, value of e_{PQ} changes with the change in the values of l,m,n. We shall determine the direction (l,m,n) for which the strain e_{PQ} is extremum (i.e. max. or min.) and we shall determine the corresponding extremum values.

e_{PQ} is a function of l,m,n of which one of the variables say n is not independent and connected with other variables by the relation $l^2+m^2+n^2=1$ ... (2)

Taking l and m as independent variable, differentiating (2) w.r.t. l and m, we get $2l + 2n\frac{\partial n}{\partial l} = 0$ ... (3)

and $2m + 2n\frac{\partial n}{\partial m} = 0$ ... (4)

Differentiating (1) w.r.t. l and m separately, ~~we get~~ and putting $\frac{\partial e_{PQ}}{\partial l} = 0 = \frac{\partial e_{PQ}}{\partial m}$ for extremum values, we get-

$$0 = 2le_{xx} + 2me_{xy} + 2ne_{zx} + 2\frac{\partial n}{\partial l}(le_{zx} + me_{yz} + ne_{zz}) \quad \text{... (5)}$$

$$0 = 2le_{xy} + 2me_{yy} + 2n e_{yz} + 2\frac{\partial n}{\partial m}(le_{zx} + me_{yz} + ne_{zz}) \quad \text{... (6)}$$

substituting for $\frac{\partial n}{\partial l}$ and $\frac{\partial n}{\partial m}$ from (3) and (4) in (5) and (6),

$$\frac{le_{xx} + me_{xy} + ne_{zx}}{l} = \frac{le_{xy} + me_{yy} + ne_{yz}}{m} = \frac{le_{zx} + me_{yz} + ne_{zz}}{n} = \epsilon \text{ (say)} \quad \text{... (7)}$$

Eq^n (7) can be written as

$$\left.\begin{aligned} l(e_{xx}-\epsilon) + me_{xy} + ne_{zx} &= 0 \\ le_{xy} + m(e_{yy}-\epsilon) + ne_{yz} &= 0 \\ le_{zx} + me_{yz} + n(e_{zz}-\epsilon) &= 0 \end{aligned}\right\} \text{ ... (8)}$$

multiplying the 1st eq^n of (8) by l, 2nd by m and 3rd by n and adding, we get

$$l^2e_{xx} + m^2e_{yy} + n^2e_{zz} + 2lme_{xy} + 2mne_{yz} + 2nle_{zx} = \epsilon(l^2+m^2+n^2) = \epsilon \;[\because l^2+m^2+n^2=1]$$

This means that values l,m,n determined from (7), give the directions along which the extension per unit length is extremum and further the corresponding value of ϵ is equal to this extremum.

In order that, a non-trivial soln of d-cs. l, m, n from eqn (8) may exist the determinant of the co-efficient should be zero i.e.

$$\begin{vmatrix} e_{xx}-\epsilon & e_{xy} & e_{xz} \\ e_{xy} & e_{yy}-\epsilon & e_{yz} \\ e_{xz} & e_{zy} & e_{zz}-\epsilon \end{vmatrix} = 0 \quad \cdots (9)$$

expanding the determinant (9) results,

$$\epsilon^3 - \mathcal{J}_1 \epsilon^2 + \mathcal{J}_2 \epsilon - \mathcal{J}_3 = 0 \quad \cdots (10) \qquad [\because \mathcal{J}_1 = J]$$

where $\mathcal{J}_1 = e_{xx} + e_{yy} + e_{zz}$

$$\mathcal{J}_2 = \begin{vmatrix} e_{xx} & e_{xy} \\ e_{xy} & e_{yy} \end{vmatrix} + \begin{vmatrix} e_{xx} & e_{xz} \\ e_{zx} & e_{zz} \end{vmatrix} + \begin{vmatrix} e_{yy} & e_{yz} \\ e_{zy} & e_{zz} \end{vmatrix}$$

$$\mathcal{J}_3 = \begin{vmatrix} e_{xx} & e_{xy} & e_{xz} \\ e_{xy} & e_{yy} & e_{yz} \\ e_{xz} & e_{yz} & e_{zz} \end{vmatrix}$$

The three roots of eqn (9), viz, $\epsilon_1, \epsilon_2, \epsilon_3$ are known as principal strains and the corresponding values of l, m, n determined from (8) which are (l_1, m_1, n_1); (l_2, m_2, n_2) and (l_3, m_3, n_3) are called the principal directions of strain and lines through $p(x, y, z)$ in these directions principal axes of strains at P.

Since the principal strain values $\epsilon_1, \epsilon_2, \epsilon_3$ at a point are independent of the choice of the co-ordinate axes, so $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3$ being equal to $\epsilon_1+\epsilon_2+\epsilon_3$, $\epsilon_1\epsilon_2+\epsilon_2\epsilon_3+\epsilon_3\epsilon_1$ and $\epsilon_1\epsilon_2\epsilon_3$ respectively are independent of the choice of the co-ordinate axes and are therefore invariant. 280

$\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3$ are respectively called the 1st, 2nd and 3rd invariants.

Now we shall prove that, principal axes of strain at any point are mutually orthogonal.

If l_1, m_1, n_1 be the one of the directions of principal strains at $P(x, y, z)$ and ϵ_1 be the corresponding value of principal strain then from (8)

$$\left.\begin{aligned} l_1 e_{xx} + m_1 e_{xy} + n_1 e_{zx} &= \epsilon_1 l_1 \\ l_1 e_{xy} + m_1 e_{yy} + n_1 e_{zy} &= \epsilon_1 m_1 \\ l_1 e_{xz} + m_1 e_{yz} + n_1 e_{zz} &= \epsilon_1 n_1 \end{aligned}\right\} \cdots (1)$$

similarly, if l_2, m_2, n_2 be the another principal strain direction with ϵ_2 be the corresponding value of principal strain then.

$$\left.\begin{aligned} l_2 e_{xx} + m_2 e_{xy} + n_2 e_{zx} &= \epsilon_2 l_2 \\ l_2 e_{xy} + m_2 e_{yy} + n_2 e_{yz} &= \epsilon_2 m_2 \\ l_2 e_{xz} + m_2 e_{yz} + n_2 e_{zz} &= \epsilon_2 n_2 \end{aligned}\right\} \quad \text{--- (12)}$$

multiplying 1st, 2nd and 3rd equations of (11) by l_2, m_2, n_2 respectively and adding,

$$l_1 l_2 e_{xx} + m_1 m_2 e_{yy} + n_1 n_2 e_{zz} + (m_2 l_1 + m_1 l_2) e_{xy}$$

$$+ (l_2 n_1 + n_2 l_1) e_{xz} + (n_1 m_2 + n_2 m_1) e_{yz} = \epsilon_1 (l_1 l_2 + m_1 m_2 + n_1 n_2)$$

multiplying 1st, 2nd and 3rd eq^n of (12) by l_1, m_1, n_1 respt. and adding, we get --- (13)

$$l_1 l_2 e_{xx} + m_1 m_2 e_{yy} + n_1 n_2 e_{zz} + (m_2 l_1 + m_1 l_2) e_{xy}$$

$$+ (l_2 n_1 + l_1 n_2) e_{xz} + (m_2 n_1 + n_1 m_2) e_{yz} = \epsilon_2 (l_1 l_2 + m_1 m_2 + n_1 n_2)$$

subtracting this two eq^n (13) and (14), we obtain --- (15)

$$(\epsilon_1 - \epsilon_2)(l_1 l_2 + m_1 m_2 + n_1 n_2) = 0$$

If $\epsilon_1 \neq \epsilon_2 \Rightarrow l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$

i.e. principal strain directions (l_1, m_1, n_1) and (l_2, m_2, n_2) are perpendicular.

similarly, it can be shown that principal strain directions l_3, m_3, n_3 is also perpendicular to (l_1, m_1, n_1) and (l_2, m_2, n_2). Therefore three principal strain directions are mutually perpendicular.

If the principal strain directions (l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3) are taken as the d-cs of x', y', z' axes respectively, then

$$e_{x'y'} = l_1 l_2 e_{xx} + m_1 m_2 e_{yy} + n_1 n_2 e_{zz} + (l_1 m_2 + l_2 m_1) e_{xy}$$

$$+ (m_1 n_2 + n_1 m_1) e_{yz} + (l_1 n_2 + l_2 n_1) e_{zx} = 0 \quad [\text{by (13) and (15)}]$$

similarly, $e_{y'z'} = e_{z'x'} = 0$

Therefore, if at any point principal axes of strains are taken as the axes of refference then with referrence to these set of axes, shearing stress are zero.

28.2.96

Compatibility conditions for infinitesimal strain:

Let Ox_1, Ox_2, Ox_3 be three orthogonal axes and let u_1, u_2, u_3 be the components of displacement at any point in the direction of x_1, x_2, x_3 axes respectively. When the three displacement functions u_1, u_2, u_3 are given we can always find the six strain compts. in any region where the partial derivatives $\frac{\partial u_i}{\partial x_j}$ exists. On the other hand when the six strain components e_{ij} are arbitrarily prescribed in some region, in general, there may not exist any displacement field u_i satisfying the equations $\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = e_{ij}$; $i,j = 1,2,3$

$u_x = u_1$

$u_y = u_2$

$u_z = u_3$

$\frac{\partial u_x}{\partial x} = \frac{\partial u_1}{\partial x_1}$

$\frac{\partial u_i}{\partial x_j}$ ie

Ex! If we set $e_{11} = x_2^2$, $e_{22} = e_{33} = e_{12} = e_{23} = e_{31} = 0$

then from $e_{11} = \frac{\partial u_1}{\partial x_1} = x_2^2$, $e_{22} = \frac{\partial u_2}{\partial x_2} = 0$

Int., $u_1 = x_2^2 x_1 + f(x_2, x_3)$ and $u_2 = g(x_1, x_3)$

where f and g are arbitrary functions.

$\because$ $e_{12} = 0$, so we must have, $\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = 0$

or, $2x_1x_2 + \frac{\partial}{\partial x_2} f(x_2, x_3) + \frac{\partial g(x_1, x_3)}{\partial x_1} = 0$

$\because$ the second and third terms can not have terms of the form $x_1 x_2$, the above equation can never be satisfied.

In other words there is no displacement field corresponding to the given e_{ij}. We therefore say that the given e_{ij}'s are not compatible.

We shall prove that if e_{ij} are cont. functions having second (order) partial derivatives in a simply connected region then the necessary and sufficient conditions for the existence of single valued continuous solutions u_i of equations

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$ are

(i) $$\frac{\partial^2 e_{11}}{\partial x_2^2} + \frac{\partial^2 e_{22}}{\partial x_1^2} = 2\frac{\partial^2 e_{12}}{\partial x_1 \partial x_2}$$

(ii) $$\frac{\partial^2 e_{22}}{\partial x_3^2} + \frac{\partial^2 e_{33}}{\partial x_2^2} = 2\frac{\partial^2 e_{23}}{\partial x_2 \partial x_3}$$

(iii) $$\frac{\partial^2 e_{33}}{\partial x_1^2} + \frac{\partial^2 e_{11}}{\partial x_3^2} = 2\frac{\partial^2 e_{31}}{\partial x_1 \partial x_3}$$

(iv) $$\frac{\partial^2 e_{11}}{\partial x_2 \partial x_3} = \frac{\partial}{\partial x_1}\left(-\frac{\partial e_{23}}{\partial x_1} + \frac{\partial e_{31}}{\partial x_2} + \frac{\partial e_{12}}{\partial x_3}\right)$$

(v) $\dfrac{\partial^2 e_{22}}{\partial x_3 \partial x_1} = \dfrac{\partial}{\partial x_2}\left(-\dfrac{\partial e_{31}}{\partial x_2} + \dfrac{\partial e_{12}}{\partial x_3} + \dfrac{\partial}{\partial x_1} e_{23}\right)$

(vi) $\dfrac{\partial^2 e_{33}}{\partial x_1 \partial x_2} = \dfrac{\partial}{\partial x_3}\left(-\dfrac{\partial}{\partial x_3} e_{12} + \dfrac{\partial e_{23}}{\partial x_1} + \dfrac{\partial e_{31}}{\partial x_2}\right)$

These six equations are known as <u>the equations of compatibility</u>.

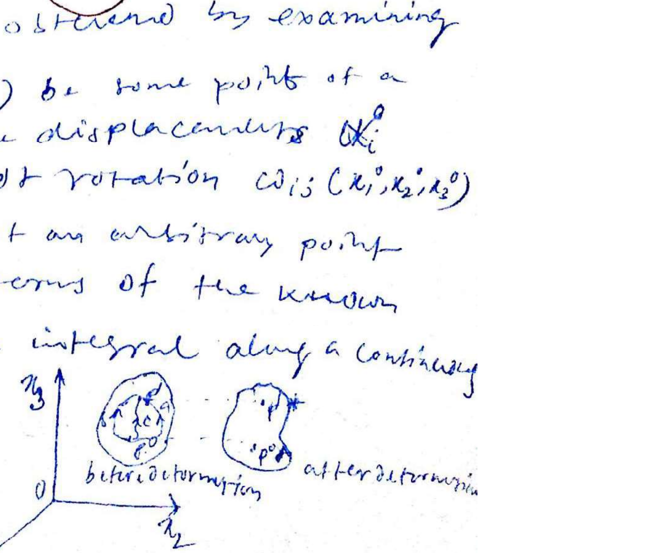
<u>Necessary and sufficient condition for strain components to give single valued displacements for a simply connected region:</u>

We have defined the components of linear strains in terms of the components of the displacement by $e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ -- (A)

If the displacement components u_1, u_2, u_3 are prescribed continuous functions of x_1, x_2, x_3 then the strain components can be uniquely determined. On the other hand if the strain components are prescribed functions of the co-ordinates, it will not be possible to find unique values for the displacement because <u>the strains represent pure deformation ~~and rigid body~~ whereas the displacement include both deformation and rigid body motion.</u>

This difficulty is overcome by specifying the rigid body motion of a point of the body by specifying its displacement u_i and the elements of its rotation ω_{ij}. The strain displacement relations (A) form a system of six partial differential equations with only three unknowns u_1, u_2, u_3; it is obvious that some restrictions must be placed on the strains in order that equations (A) have a solution. These restrictions are called <u>compatibility relations</u>. (2 95) 2 92 2 89

Conditions of compatibility can be obtained by examining a deformed body. Let $P^0(x_1^0, x_2^0, x_3^0)$ be some point of a simply connected region at which the displacements u_i^0 ($u_i^0(x_1^0, x_2^0, x_3^0)$) and the components of rotation $\omega_{ij}(x_1^0, x_2^0, x_3^0)$ are known. The displacement u_i of an arbitrary point $P'(x_1', x_2', x_3')$ can be obtained in terms of the known function e_{ij} by means of a line integral along a continuous curve C joining P^0 and P'



Thus $u_i(x_1', x_2', x_3') = u_i^0 + \displaystyle\int_{P^0}^{P'} du_i$ -- (1)

If in the process of deformation the body remains continuous u_i should be independent of the path of integration i.e. u_i should have the same value regardless of whether the integration along the curve a, b, c or any other path.

From ①, we have
$$u_i(x_1', x_2', x_3') = u_i^0 + \oint_{p^0}^{p'} u_{i,j}\, dx_j$$

$$\left[\text{where } du_i = \frac{\partial u_i}{\partial x_j} dx_j = u_{i,j}\, dx_j\right]$$

$$= u_i^0 + \oint_{p^0}^{p'} \left[\tfrac{1}{2}(u_{i,j}+u_{j,i}) + \tfrac{1}{2}(u_{i,j}-u_{j,i})\right] dx_j$$

$$= u_i^0 + \oint_{p^0}^{p'} e_{ij}\, dx_j + \int_{p^0}^{p'} \omega_{ij}\, dx_j$$

Integrating by parts the last integral we get

$$u_i(x_1', x_2', x_3') = u_i^0 + \oint_{p^0}^{p'} e_{ij}\, dx_j + [\omega_{ij} x_j]_{p^0}^{p'} - \int_{p^0}^{p'} x_j\, \omega_{ij,k}\, dx_k$$

$$\left[\omega_{ij,k} = \frac{\partial \omega_{ij}}{\partial x_k}\right]$$

$$= u_i^0 + [\omega_{ij} x_j]_{p^0}^{p'} + \oint_{p^0}^{p'} [e_{ik} - x_j \omega_{ij,k}]\, dx_k$$

Now
$$\omega_{ij,k} = \frac{\partial}{\partial x_k}\left[\tfrac{1}{2}(u_{i,j} - u_{j,i})\right]$$

$$= \tfrac{1}{2}(u_{i,jk} - u_{j,ik}) + \tfrac{1}{2}(u_{k,ji} - u_{k,ji})$$

$$= \frac{\partial}{\partial x_j}\left[\tfrac{1}{2}(u_{i,k}+u_{k,i})\right] - \frac{\partial}{\partial x_i}\left[\tfrac{1}{2}(u_{j,k}+u_{k,j})\right]$$

$$= e_{ik,j} - e_{jk,i}$$

Therefore
$$u_i(x_1', x_2', x_3') = u_i^0 + [\omega_{ij} x_j]_{p^0}^{p'} + \oint_{p^0}^{p'} [e_{ik} - x_j(e_{ik,j} - e_{jk,i})]\, dx_k$$

Let
$$U_{ik} = e_{ik} - x_j(e_{ik,j} - e_{jk,i}) \qquad \text{--- ②}$$

The 1st two terms on the R.H.S. of ①, are independent of the path of integration. ~~Now~~ For the third term to be independent of the path, the integral

$U_{ik}\, dx_k$ must be an exact differential, the condition for which is :
$$\frac{\partial U_{ik}}{\partial x_l} = \frac{\partial U_{il}}{\partial x_k}$$

Now
$$\frac{\partial U_{ik}}{\partial x_l} = e_{ik,l} - x_j(e_{ik,jl} - e_{jk,il}) - \delta_{jl}(e_{ik,j} - e_{jk,l})$$

$$\frac{\partial \omega_{il}}{\partial x_k} = e_{il,k} - x_j\left(e_{il,jk} - e_{jl,ik}\right) - \delta_{jk}\left(e_{il,j} - e_{jl,i}\right)$$

$$0 = \frac{\partial \omega_{ik}}{\partial x_l} - \frac{\partial \omega_{il}}{\partial x_k} = \Big[e_{ik,l} - \delta_{jl}\left(e_{ik,j} - e_{jk,l}\right)$$

$$\cancel{+0} - e_{il,k} + \delta_{jk}\left(e_{il,j} - e_{jl,i}\right)\Big]$$

$$+ x_j\left(-e_{ik,jl} - e_{jl,ik} + e_{jk,il} + e_{il,jk}\right)$$

the quantity with in the 3rd bracket on the R.H.S is zero identically so since x_j''s are independent, the necessary and sufficient condition that $u_i'(x_1', x_2', x_3')$ may be independent of the path of integration are

$$e_{jk,il} + e_{il,jk} - e_{ik,jl} - e_{jl,ik} = 0 \quad \cdots \quad (3)$$

These are the compatibility relations. Although eq^n (3) results in 81 eq^{n}s on account of its four different subscripts, only six deserve consideration. The others are identically satisfied or are repetations resulting from the symmetry of e_{ij}. In details, from (3), we get, by putting

(i) ~~Let~~ $j = 1, k = 2, i = 2, l = 1$

$$2\frac{\partial^2 e_{12}}{\partial x_1 \partial x_2} = \frac{\partial^2 e_{11}}{\partial x_2^2} + \frac{\partial^2 e_{22}}{\partial x_1^2}$$

(ii) $j = 2, k = 3, i = 3, l = 2$

$$2\frac{\partial^2 e_{23}}{\partial x_2 \partial x_3} = \frac{\partial^2 e_{22}}{\partial x_3^2} + \frac{\partial^2 e_{33}}{\partial x_2^2}$$

(iii) $j = 3, k = 1, i = 1, l = 3$

$$2\frac{\partial^2 e_{31}}{\partial x_1 \partial x_3} = \frac{\partial^2 e_{33}}{\partial x_1^2} + \frac{\partial^2 e_{11}}{\partial x_3^2}$$

(iv) $j = 1, k = 1, i = 2, l = 3$

$$\frac{\partial^2 e_{11}}{\partial x_2 \partial x_3} = \frac{\partial}{\partial x_1}\left(\frac{\partial e_{12}}{\partial x_3} + \frac{\partial e_{13}}{\partial x_2} - \frac{\partial e_{23}}{\partial x_1}\right)$$

(v) $j = 2, k = 2, i = 3, l = 1$

$$\frac{\partial^2 e_{22}}{\partial x_3 \partial x_1} = \frac{\partial}{\partial x_2}\left(\frac{\partial e_{23}}{\partial x_1} + \frac{\partial e_{12}}{\partial x_3} - \frac{\partial e_{13}}{\partial x_2}\right)$$

(vi) $j = 3, k = 3, i = 1, l = 2$, $\quad \frac{\partial^2 e_{33}}{\partial x_1 \partial x_2} = \frac{\partial}{\partial x_3}\left(\frac{\partial e_{13}}{\partial x_2} + \frac{\partial e_{23}}{\partial x_1} - \frac{\partial e_{12}}{\partial x_3}\right)$

CHAP: 3

CONSTITUTIVE EQUATIONS

An equation which describes a property of a material is called a Constitutive equation of that material. A stress strain relationship describes the mechanical property of a material and is therefore a constitutive equation.

Constitutive equation for elastic material:

Hooke's Law: If an elastic body has a simple tension τ_{xx} in the x-direction producing thereby a longitudinal strain e_{xx} in the same direction within elastic limit then Hooke's law is expressed mathematically as

$$\frac{\tau_{xx}}{e_{xx}} = E,$$ where E is the Young's modulus of the material. So $\tau_{xx} = E.e_{xx}$, this idea has been generalised in a law known as generalised Hooke's law. We shall assume that there is one to one analytic relation $\tau_{ij} = F_{ij}(e_{xx}, e_{yy}, e_{zz}, e_{xy}, e_{yz}, e_{zx})$ between τ_{ij} and F_{ij} and τ_{ij} vanish when the strains e_{ij} are all zero. Now if the function F_{ij} are expanded in power series in e_{ij} and only the linear terms are retained, we obtain

$$\tau_{xx} = c_{11}e_{xx} + c_{12}e_{yy} + c_{13}e_{zz} + c_{14}e_{yz} + c_{15}e_{zx} + c_{16}e_{xy}$$

$$\tau_{yy} = c_{21}e_{xx} + c_{22}e_{yy} + c_{23}e_{zz} + c_{24}e_{yz} + c_{25}e_{zx} + c_{26}e_{xy}$$

$$\tau_{zz} = c_{31}e_{xx} + c_{32}e_{yy} + c_{33}e_{zz} + c_{34}e_{yz} + c_{35}e_{zx} + c_{36}e_{xy}$$

$$\tau_{yz} = c_{41}e_{xx} + c_{42}e_{yy} + c_{43}e_{zz} + c_{44}e_{yz} + c_{45}e_{zx} + c_{46}e_{xy}$$

$$\tau_{zx} = c_{51}e_{xx} + c_{52}e_{yy} + c_{53}e_{zz} + c_{54}e_{yz} + c_{55}e_{zx} + c_{56}e_{xy}$$

$$\tau_{xy} = c_{61}e_{xx} + c_{62}e_{yy} + c_{63}e_{zz} + c_{64}e_{yz} + c_{65}e_{zx} + c_{66}e_{xy}$$

The coefficients $c_{11}, c_{12}, \ldots$ etc. will in general vary point to point of the medium. If the coefficients are independent of the position of the point the medium is called elastically homogeneous. A homogeneous material may again become isotropic or anisotropic. "For an isotropic body the elastic property of the body are identical in all directions whereas for an anisotropic body the elastic properties become different in different directions."